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PERFORMANCE OF THE ESTIMATORS OF STABLE LAW PARAMETERS

Abstract: In this paper, we discuss the issue of estimation of the parameters of stable laws. We present an overview of the known methods and compare them on samples of different sizes and for different values of the parameters. Performance tables are provided.

1 Introduction

The Central Limit Theorem, which offers the fundamental justification for approximate normality, points to the importance of α–stable (sometimes called stable) distributions: they are the only limiting laws of normalized sums of independent, identically distributed random variables. Gaussian distributions, the best known member of the stable family, have long been well understood and widely used in all sorts of problems. However, they do not allow for large fluctuations and are thus inadequate for modeling high variability. Non-Gaussian stable models, on the other hand, do not share such limitations. In general, the upper and lower tails of their distributions decrease like a power function. In literature, this is often characterized as heavy or long tails. In the last two or three decades, data which seem to fit the stable model have been collected in fields as diverse as economics, telecommunications, hydrology and physics.

This paper is divided into sections which, we hope, will guide the reader from theory, through simulation to estimation of parameters. Section 2 is an introduction to the stable family. We explain the differences between the most often used, in literature, representations of the skewed stable characteristic function and the confusion around it. Section 3 is a guide to simulation of α–stable random variables. The equality in law of a skewed stable variable and a function of two independent uniform and exponential variables (Theorem 3.1) is discussed. The section is closed by a discussion of some minor errors in this formula found in different publications. Section 4 is concerned with the issue of estimation of the parameters of stable laws. We present an overview of the known methods and compare them on samples of different sizes and for different values of the parameters. Basing on the results of Section 3 we are able to compare the performance of four most often used (Fama–Roll's, McCulloch's, moments and regression) estimators of stable law parameters. Throughout the paper, we have tried to make this exciting material easily accessible to researchers and practitioners. We hope that we have accomplished this.

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2 α–Stable Distribution

The notion of stable probability law was introduced by Lévy (1924) during his early investigations of the behavior of sums of independent random variables. The ambiguous name *stable* has been assigned to these distributions because a sum of two independent random variables having a stable distribution with index α is again stable with the same index α.

The stable distribution can be most conveniently described by its *characteristic function* (cf). The following formula is derived from the so–called Lévy representation of the cf of an infinitely divisible law, given in Lévy (1934) (for details see Hall (1981)).

Definition 2.1 *A random variable X is α–stable if and only if its characteristic function is given by*

$$\log \phi(t) = \begin{cases} -\sigma^\alpha |t|^\alpha \{1 - i\beta sign(t) \tan \frac{\pi\alpha}{2}\} + i\mu t, & \alpha \neq 1, \\ \\ -\sigma|t|\{1 + i\beta sign(t) \frac{2}{\pi} \log|t|\} + i\mu t, & \alpha = 1, \end{cases} \tag{2.1}$$

where $\alpha \in (0,2]$, $\beta \in [-1,1]$, $\sigma > 0$, $\mu \in R$.

Since (2.1) is characterized by four parameters we will denote α–stable distributions by $S_\alpha(\sigma,\beta,\mu)$ and write

$$X \sim S_\alpha(\sigma,\beta,\mu) \tag{2.2}$$

to indicate that X has the stable distribution with the *characteristic exponent* (index) α, *scale parameter* σ, *skewness* β and *location parameter* μ. When $\sigma = 1$ and $\mu = 0$ the distribution is called *standard stable*.

Some authors use a form similar to (2.1), but with the sign on the term involving β reversed for $\alpha \neq 1$. This "β" is positive (negative) when the distribution is negatively (positively) skewed, except when $\alpha = 1$. This confusing convention was used in many important papers, including Press (1972a, 1972b), Paulson et al. (1975), Leitch and Paulson (1975) and Koutrouvelis (1980, 1981).

The canonical representation (2.1) has one disagreeable feature. The functions $\phi(t)$ are not continuous functions of the parameters determining them, they have discontinuities at all points of the form $\alpha = 1, \beta \neq 0$. However, as Zolotarev (1986) remarks, setting

$$\mu_1 = \begin{cases} \mu + \beta\sigma^\alpha \tan\frac{\pi\alpha}{2}, & \alpha \neq 1, \\ \\ \mu, & \alpha = 1, \end{cases} \tag{2.3}$$

yields the expression

$$\log \phi(t) = \begin{cases} -\sigma^\alpha \{|t|^\alpha - it\beta(|t|^{\alpha-1} - 1) \tan\frac{\pi\alpha}{2}\} + i\mu_1 t, & \alpha \neq 1, \\ \\ -\sigma|t|\{1 + i\beta sign(t)\frac{2}{\pi}\log|t|\} + i\mu_1 t, & \alpha = 1, \end{cases} \tag{2.4}$$

which is a function jointly continuous in α and β. The drawback of this form is that μ_1 does no longer have the natural interpretation as a location parameter. Most authors, therefore, use the form (2.1) of the cf.

Another form of the cf, whose use can be justified by considerations of an analytic nature (see Zolotarev (1986)), is the following.

Definition 2.2 *A random variable X is α–stable iff its characteristic function is given by*

$$\log \phi(t) = \begin{cases} -\sigma_2^\alpha |t|^\alpha \exp\{-i\beta_2 sign(t)\frac{\pi}{2}K(\alpha)\} + i\mu t, & \alpha \neq 1, \\ -\sigma_2|t|\{\frac{\pi}{2} + i\beta_2 sign(t)\log|t|\} + i\mu t, & \alpha = 1, \end{cases} \tag{2.5}$$

where

$$K(\alpha) = \alpha - 1 + sign(1-\alpha) = \begin{cases} \alpha, & \alpha < 1, \\ \alpha - 2, & \alpha > 1. \end{cases} \tag{2.6}$$

The parameters σ_2 and β_2 are related to σ and β, from the representation (2.1), as follows. For $\alpha \neq 1$, β_2 is such that

$$\tan\left(\beta_2 \frac{\pi K(\alpha)}{2}\right) = \beta \tan\frac{\pi\alpha}{2}, \tag{2.7}$$

and the new scale parameter

$$\sigma_2 = \sigma\left(1 + \beta^2\tan^2\frac{\pi\alpha}{2}\right)^{1/(2\alpha)}. \tag{2.8}$$

For $\alpha = 1$, $\beta_2 = \beta$ and $\sigma_2 = \frac{2}{\pi}\sigma$.

The *probability density functions* (pdf) of stable random variables exist and are continuous but, with a few exceptions, they are not known in closed form. The exceptions are

- *the Gaussian distribution:* $S_2(\sigma, 0, \mu) = N(\mu, 2\sigma^2)$,
- *the Cauchy distribution:* $S_1(\sigma, 0, \mu)$,
- *the Lévy distributions:* $S_{1/2}(\sigma, 1, \mu)$, $S_{1/2}(\sigma, -1, \mu)$.

3 Computer Generation of α–Stable Random Variables

The complexity of the problem of simulation of sequences of stable random variables results from the fact that there are no analytic expressions for the inverse F^{-1} of the df. The only exceptions are the Gaussian, the Cauchy and the Lévy distributions, for which simple methods of simulation have been found.

A solution of the problem was found by a path started in the article by Kanter (1975), in which a direct method was given for simulating $S_\alpha(1,1,0)$ random variables, for $\alpha < 1$. It turned out that this method was easily adopted to the general case. Chambers et al. (1976) were the first to give the formulas.

Theorem 3.1 *Let*

$$\gamma_0 = -\frac{\pi}{2}\beta_2\frac{K(\alpha)}{\alpha},$$

γ be uniformly distributed on $(-\frac{\pi}{2}, \frac{\pi}{2})$ and W be an independent exponential random variable with mean 1. Then

- *for* $\alpha \neq 1$
$$X = \frac{\sin \alpha(\gamma - \gamma_0)}{(\cos \gamma)^{1/\alpha}} \left(\frac{\cos(\gamma - \alpha(\gamma - \gamma_0))}{W} \right)^{(1-\alpha)/\alpha}, \tag{3.1}$$
is $S_\alpha(1, \beta_2, 0)$ *and*

- *for* $\alpha = 1$
$$X = (\frac{\pi}{2} + \beta_2 \gamma) \tan \gamma - \beta_2 \log \left(\frac{W \cos \gamma}{\frac{\pi}{2} + \beta_2 \gamma} \right) \tag{3.2}$$
is $S_1(1, \beta_2, 0)$

for the representation (2.5).

Proof: see Weron (1995).

Applying this theorem one can easily construct a method of computer generation of a skewed random variable $X \sim S_\alpha(1, \beta, 0)$, in the representation (2.1). For $\alpha \in (0, 2]$ and $\beta \in [-1, 1]$:

- generate a random variable V uniformly distributed on $(-\frac{\pi}{2}, \frac{\pi}{2})$ and an independent exponential random variable W with mean 1;
- for $\alpha \neq 1$ compute
$$X = S_{\alpha,\beta} \times \frac{\sin(\alpha(V + B_{\alpha,\beta}))}{(\cos(V))^{1/\alpha}} \times \left(\frac{\cos(V - \alpha(V + B_{\alpha,\beta}))}{W} \right)^{(1-\alpha)/\alpha}, \tag{3.3}$$
where
$$\begin{aligned} B_{\alpha,\beta} &= \frac{\arctan(\beta \tan \frac{\pi\alpha}{2})}{\alpha}, \\ S_{\alpha,\beta} &= \left[1 + \beta^2 \tan^2 \frac{\pi\alpha}{2} \right]^{1/(2\alpha)}; \end{aligned}$$
- for $\alpha = 1$ compute
$$X = \frac{2}{\pi} \left[(\frac{\pi}{2} + \beta V) \tan V - \beta \log \left(\frac{W \cos V}{\frac{\pi}{2} + \beta V} \right) \right]. \tag{3.4}$$

$B_{\alpha,\beta}$ accounts for the parameter change from β_2 to β and takes place of γ_0 in (3.1). $S_{\alpha,\beta}$ accounts for the parameter change from σ_2 to σ (see (2.8)).

Formula (3.3) was initially presented by Janicki and Weron (1994). However, they gave an incorrect form for $C_{\alpha,\beta}$ (the denominator is $1 - |1 - \alpha|$ instead of α, Formula (3.5.2), page 50), which corresponds to our $B_{\alpha,\beta}$, and a computationally more complicated form for $D_{\alpha,\beta}$ (our $S_{\alpha,\beta}$). They also did not provide the formula for $\alpha = 1$.

Chambers et al. (1976) give a formula ((2.3) on page 341) for $\alpha \neq 1$ equivalent to (3.3), in the representation (2.5). Their formula for $\alpha = 1$ ((2.4) on page 341) has a slightly incorrect form: under the logarithm is $\frac{\frac{\pi}{2} W \cos V}{\frac{\pi}{2} + \beta V}$ whereas it should be $\frac{W \cos V}{\frac{\pi}{2} + \beta V}$. However, this has no impact on their numerical algorithm RSTAB, since it uses a continuous representation, equivalent to (2.4), instead of (2.5).

We have given formulas for simulation of standard stable random variables. Using the following property, which follows from the form of the cf, we can simulate a stable random variable for all admissable values of the parameters α, σ, β and μ:

If $X \sim S_\alpha(1, \beta, 0)$ then

$$Y = \begin{cases} \sigma X + \mu, & \alpha \neq 1, \\ \\ \sigma X + \frac{2}{\pi}\beta\sigma \log \sigma + \mu, & \alpha = 1, \end{cases} \tag{3.5}$$

is $S_\alpha(\sigma, \beta, \mu)$.

4 Estimation of Stable Law Parameters

The problem of estimating the parameters of a stable distribution is in general severely hampered by the lack of known closed–form density functions for all but a few members of the stable family. Most of the conventional methods in mathematical statistics, including maximum likelihood estimation method, cannot be used in this case, since these methods depend on an explicit form for the density. However, there are numerical methods that have been found useful in practice and are discussed in this section.

Given a sample $x_1, ..., x_n$ from $S_\alpha(\sigma, \beta, \mu)$ we will provide estimates $\hat{\alpha}$, $\hat{\sigma}$, $\hat{\beta}$ and $\hat{\mu}$ of α, σ, β and μ respectively.

4.1 Maximum Likelihood Method

DuMouchel (1971) was the first to obtain approximate ML estimates of α and σ (assuming $\mu = 0$). A multinomial approximation to the likelihood function is used in his approach. Under some additional assumptions on $\hat{\alpha}$ and the likelihood function, DuMouchel (1973) has shown the obtained estimates to be consistent and asymptotically normal. However, the computational effort involved seems considerable.

A direct method can be formulated, after Brorsen and Yang (1990), as follows. The standard symmetric stable pdf is given by (see Zolotarev (1986), Theorem 2.2.3, page 74)

$$f_\alpha(x) = \frac{\alpha}{\pi|1-\alpha|} x^{1/(\alpha-1)} \int_0^{\pi/2} U_\alpha(\gamma, 0) e^{-x^{\alpha/(\alpha-1)} U_\alpha(\gamma,0)} d\gamma, \tag{4.1}$$

for $\alpha \neq 1, x > 0$, where U_α is defined by

$$U_\alpha(\gamma, \gamma_0) = \left(\frac{\sin \alpha(\gamma - \gamma_0)}{\cos \gamma} \right)^{\alpha/(1-\alpha)} \frac{\cos(\gamma - \alpha(\gamma - \gamma_0))}{\cos \gamma}, \tag{4.2}$$

and γ_0 is defined in Theorem 3.1. Therefore, the parameters α, σ and μ can be estimated from the observations $x_1, ..., x_n$ by maximizing the log likelihood function

$$\sum_{i=1}^{n} \log f_\alpha(z_i) \quad = \quad n \log \alpha - n \log(\alpha - 1)\pi$$

$$+ \sum_{i=1}^{n} \frac{\log z_i}{\alpha - 1}$$

$$+ \sum_{i=1}^{n} \log \int_0^{\pi/2} U_\alpha(\gamma, 0) e^{-z_i^{\alpha/(\alpha-1)} U_\alpha(\gamma,0)} d\gamma, \tag{4.3}$$

where $z_i = |x_i - \mu|/\sigma$.

To avoid the discontinuity and nondifferentiability of the symmetric α–stable density function at $\alpha = 1$, α is restricted to be greater than one. Caution must be used when evaluating the integrals (4.1) and (4.3), since the integrands are singular at $\gamma = 0$.

Based on (4.3), Brorsen and Yang (1990) performed Monte Carlo simulations with fairly good results. An obvious disadvantage of this method is that it is a highly nonlinear optimization problem and no initialization and convergence analysis is available.

4.2 Sample Quantiles Methods

Let x_f be the f–th population *quantile*, so that $S_\alpha(\sigma, \beta, \mu)(x_f) = f$. Let $\hat{x}_f$ be the corresponding *sample quantile*, i.e. $\hat{x}_f$ satisfies $F_n(\hat{x}_f) = f$. As McCulloch (1986) points out, to avoid spurious skewness in finite samples, a correction must be made. If the x_i's are arranged in ascending order, the correction may be performed by identifying x_i with $\hat{x}_{q(i)}$, where $q(i) = \frac{2i-1}{2n}$, and then interpolating linearly to f from the two adjacent $q(i)$ values. Then $\hat{x}_f$ is a consistent estimator of x_f, the f quantile.

4.2.1 Fama–Roll Method

Fama and Roll (1968, 1971) provided estimates for parameters of symmetric ($\beta = 0, \mu = 0$) stable laws with $1 < \alpha \leq 2$. They propose to estimate σ by

$$\hat{\sigma} = \frac{\hat{x}_{0.72} - \hat{x}_{0.28}}{1.654}. \tag{4.4}$$

McCulloch (1986) notices that Fama and Roll base their estimator of σ on the fortuitous observation that $(x_{0.72} - x_{0.28})/\sigma$ lies within 0.4% of 1.654 for all $1 < \alpha \leq 2$, when $\beta = 0$. This enables them to estimate σ by (4.4) with less than 0.4% asymptotic bias without first knowing α. However, when $\beta \neq 0$, the search for an invariant range such as the one they found becomes futile.

The characteristic exponent α, on the other hand, can be estimated from the tail behavior of the distribution. Fama and Roll take $\hat{\alpha}$ satisfying

$$S_{\hat{\alpha}}\left(\frac{\hat{x}_f - \hat{x}_{1-f}}{2\hat{\sigma}}\right) = f. \tag{4.5}$$

They find that $f = 0.95, 0.96, 0.97$ works best for estimating α. This method unnecessarily compounds the small asymptotic bias in the estimator of σ into the estimator of α. Tabulated values of $S_{\hat{\alpha}}$ can be found in Brothers et al. (1983) or in Samorodnitsky and Taqqu (1994).

Fama and Roll also note that since x_j's follow a stable distribution, for every p, $\sum_{i=1}^{p} x_i \sim S_\alpha(\sigma_1, 0, 0)$, where $\sigma_1 = p^{1/\alpha}\sigma$. Solving for α and replacing the parameters by their estimators gives

$$\hat{\alpha} = \frac{\log p}{\log \hat{\sigma}_1 - \log \hat{\sigma}}. \tag{4.6}$$

The choice of p quite questionable. Our preliminary simulations proved this estimator to be inferior to all other estimators of α compared in this paper and we have not studied it extensively in Section 4.4.

For $1 < \alpha \leq 2$, the stable distribution has finite mean. Hence, the sample mean is a consistent estimate of the location parameter μ. A more robust estimate is the *p percent truncated sample mean* – the arithmetic mean of the middle p percent of the ranked observations. Numerous simulations proved the 50% truncated mean to work well when the range of α is unknown.

4.2.2 McCulloch's Method

Fama–Roll's method is simple but suffers from a small asymptotic bias in $\hat{\alpha}$ and $\hat{\sigma}$ and restrictions on α and β. McCulloch (1986) generalized and improved this method. He provided consistent estimators of all four parameters, with $0.6 \leq \alpha \leq 2$, while retaining the computational simplicity of Fama–Roll's method.

After McCulloch, define

$$v_\alpha = \frac{x_{0.95} - x_{0.05}}{x_{0.75} - x_{0.25}}, \tag{4.7}$$

which is independent of both σ and μ. Let $\hat{v}_\alpha$ be the corresponding sample value. It is a consistent estimator of v_α.

Define

$$v_\beta = \frac{x_{0.95} + x_{0.05} - 2x_{0.50}}{x_{0.95} - x_{0.05}}, \tag{4.8}$$

and let $\hat{v}_\beta$ be the corresponding sample value. v_β is also independent of both σ and μ. As a function of α and β it is strictly increasing in β for each α. The statistic $\hat{v}_\beta$ is a consistent estimator of v_β.

v_α and v_β are functions of α and β. This relationship may be inverted and the parameters α and β may be viewed as functions of v_α and v_β

$$\alpha = \psi_1(v_\alpha, v_\beta), \qquad \beta = \psi_2(v_\alpha, v_\beta). \tag{4.9}$$

Substituting v_α and v_β by their sample values yields estimators $\hat{\alpha}$ and $\hat{\beta}$.

Table 4.1 shows α as a function of v_α and v_β. Note that $\psi_1(v_\alpha, -v_\beta) = \psi_1(v_\alpha, v_\beta)$. With finite samples, it is possible that $\hat{v}_\alpha$ may be less than its smallest permissible value of 2.439, and therefore be offscale in this table. In this case, $\hat{\alpha}$ should be set equal to 2.0 and $\hat{\beta}$ may be set to 0.0.

Table 4.2 shows β as a function of v_α and v_β. Note that $\psi_2(v_\alpha, -v_\beta) = -\psi_2(v_\alpha, v_\beta)$. Entries in this table greater then 1.0 are required in order to permit accurate bivariate linear interpolation. If the interpolated estimate of β is greater then 1.0, the estimate should be truncated to 1.0.

Table 4.3 shows the behavior of

$$v_\sigma = \frac{x_{0.75} - x_{0.25}}{\sigma}, \tag{4.10}$$

as a function $\phi_3(\alpha, \beta)$. Since $\hat{\alpha}$, $\hat{\beta}$, $\hat{x}_{0.75}$ and $\hat{x}_{0.25}$ are all consistent estimators of their corresponding population values, a consistent estimator of σ is

$$\hat{\sigma} = \frac{\hat{x}_{0.75} - \hat{x}_{0.25}}{\phi_3(\hat{\alpha}, \hat{\beta})}. \tag{4.11}$$

v_α	v_β						
	0.0	0.1	0.2	0.3	0.5	0.7	1.0
2.439	2.000	2.000	2.000	2.000	2.000	2.000	2.000
2.5	1.916	1.924	1.924	1.924	1.924	1.924	1.924
2.6	1.808	1.813	1.829	1.829	1.829	1.829	1.829
2.7	1.729	1.730	1.737	1.745	1.745	1.745	1.745
2.8	1.664	1.663	1.663	1.668	1.676	1.676	1.676
3.0	1.563	1.560	1.553	1.548	1.547	1.547	1.547
3.2	1.484	1.480	1.471	1.460	1.448	1.438	1.438
3.5	1.391	1.386	1.378	1.364	1.337	1.318	1.318
4.0	1.279	1.273	1.266	1.250	1.210	1.184	1.150
5.0	1.128	1.121	1.114	1.101	1.067	1.027	0.973
6.0	1.029	1.021	1.014	1.004	0.974	0.935	0.874
8.0	0.896	0.892	0.887	0.883	0.855	0.823	0.769
10.0	0.818	0.812	0.806	0.801	0.780	0.756	0.691
15.0	0.698	0.695	0.692	0.689	0.676	0.656	0.595
25.0	0.593	0.590	0.588	0.586	0.579	0.563	0.513

Table 4.1: $\alpha = \psi_1(v_\alpha, v_\beta) = \psi_1(v_\alpha, -v_\beta)$.

v_α	v_β						
	0.0	0.1	0.2	0.3	0.5	0.7	1.0
2.439	0.000	2.160	1.000	1.000	1.000	1.000	1.000
2.5	0.000	1.592	3.390	1.000	1.000	1.000	1.000
2.6	0.000	0.759	1.800	1.000	1.000	1.000	1.000
2.7	0.000	0.482	1.048	1.694	1.000	1.000	1.000
2.8	0.000	0.360	0.760	1.232	2.229	1.000	1.000
3.0	0.000	0.253	0.518	0.823	1.575	1.000	1.000
3.2	0.000	0.203	0.410	0.632	1.244	1.906	1.000
3.5	0.000	0.165	0.332	0.499	0.943	1.560	1.000
4.0	0.000	0.136	0.271	0.404	0.689	1.230	2.195
5.0	0.000	0.109	0.216	0.323	0.539	0.827	1.917
6.0	0.000	0.096	0.190	0.284	0.472	0.693	1.759
8.0	0.000	0.082	0.163	0.243	0.412	0.601	1.596
10.0	0.000	0.074	0.147	0.220	0.377	0.546	1.482
15.0	0.000	0.064	0.128	0.191	0.330	0.478	1.362
25.0	0.000	0.056	0.112	0.167	0.285	0.428	1.274

Table 4.2: $\beta = \psi_2(v_\alpha, v_\beta) = -\psi_2(v_\alpha, -v_\beta)$.

McCulloch also gives an estimate of μ. However, due to the discontinuity of the cf at $\alpha = 1, \beta \neq 0$ in the representation (2.1), this procedure is much more complicated. We refer the interested reader to the original paper by McCulloch (1986). To illustrate the use of the presented method let us consider the following example.

Example 4.1 Analysis of series B from Box and Jenkins (1976). *Denote by $\{z_t\}$ the realized daily IBM stock prices for the period 17th May 1961 – 2nd November 1962 (369 entries). The analysis of $\{z_t\}$ is often based on $x_t = 100(\log z_t - \log z_{t-1})$, the one day percentage return under continuous compounding. See Fig. 4.1.*

We can easily compute the quantiles of $\{x_t\}$

$$\begin{aligned} \hat{x}_{0.05} &= -3.20168, \\ \hat{x}_{0.50} &= 0.00000, \\ \hat{x}_{0.95} &= 2.54767, \end{aligned}$$

α	β				
	0.0	0.25	0.5	0.75	1.0
2.0	1.908	1.908	1.908	1.908	1.908
1.9	1.914	1.915	1.916	1.918	1.921
1.8	1.921	1.922	1.927	1.936	1.947
1.7	1.927	1.930	1.943	1.961	1.987
1.6	1.933	1.940	1.962	1.997	2.043
1.5	1.939	1.952	1.988	2.045	2.116
1.4	1.946	1.967	2.022	2.106	2.211
1.3	1.955	1.984	2.067	2.188	2.333
1.2	1.965	2.007	2.125	2.294	2.491
1.1	1.980	2.040	2.205	2.435	2.696
1.0	2.000	2.085	2.311	2.624	2.973
0.9	2.040	2.149	2.461	2.886	3.356
0.8	2.098	2.244	2.676	3.265	3.912
0.7	2.189	2.392	3.004	3.844	4.775
0.6	2.337	2.635	3.542	4.808	6.247
0.5	2.588	3.073	4.534	6.636	9.144

Table 4.3: $v_\sigma = \phi_3(\alpha, \beta) = \phi_3(\alpha, -\beta)$.

$$\hat{x}_{0.75} - \hat{x}_{0.25} \quad = \quad 1.57573.$$

Then using (4.7) and (4.8) we can calculate $\hat{v}_\alpha = 3.64870$ *and* $\hat{v}_\beta = -0.11375$. *Linear interpolation to these values on Tables 4.1 and 4.2 yields*

$$\hat{\alpha} = 1.35133 \quad \textit{and} \quad \hat{\beta} = -0.17803.$$

In turn, linear interpolation to $\hat{\alpha}$ *and* $\hat{\beta}$ *on Table 4.3 gives* $\hat{v}_\sigma = 1.96811$. *Thus, from (4.11),*

$$\hat{\sigma} = 0.80063.$$

4.3 Sample Characteristic Function Methods

Given an i.i.d. random sample $x_1, ..., x_n$ of size n, define the *sample cf* by

$$\hat{\phi}(t) = \frac{1}{n} \sum_{j=1}^{n} e^{itx_j}. \tag{4.12}$$

Since, $|\hat{\phi}(t)|$ is bounded by unity all moments of $\hat{\phi}(t)$ are finite, and for any fixed t, it is the sample average of i.i.d. random variables $\exp(itx_j)$. Hence, by the law of large numbers, $\hat{\phi}(t)$ is a consistent estimator of the cf $\phi(t)$.

Furthermore, note that in the symmetric case ($\beta = 0, \mu = 0$) the cf is real and the sample cf takes the form

$$\hat{\phi}(t) = \frac{1}{n} \sum_{i=1}^{n} \cos tx_i. \tag{4.13}$$

4.3.1 Method of Moments

Press (1972a, 1972b) proposed a simple estimation method based on transformations of the cf. From (2.1) we have for all α

$$|\phi(t)| = \exp(-\sigma^\alpha |t|^\alpha). \tag{4.14}$$

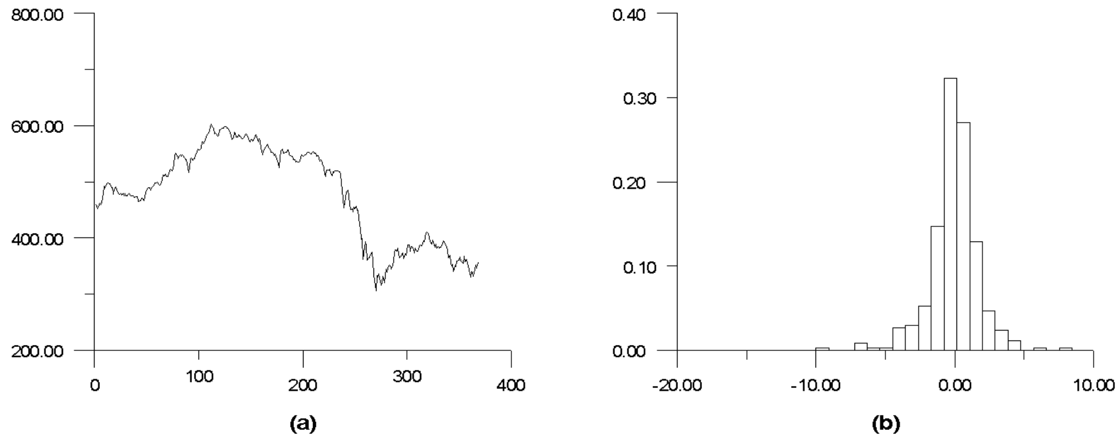

Figure 4.1: *(a) Realization of the process $\{z_t\}$: plot of the daily value of IBM stocks in \$ against session number (1 - 17.05.61, 369 - 2.11.62); (b) 20 bin histogram of $\{x_t\}$.*

Hence, $-\log|\phi(t)| = \sigma^\alpha |t|^\alpha$.

Case of $\alpha \neq 1$. Choose two nonzero values of t, $t_1 \neq t_2$. Then

$$-\log|\phi(t_k)| = \sigma^\alpha |t_k|^\alpha, \tag{4.15}$$

for $k = 1, 2$. Solving these two equations simultaneously for α and σ, and substituting $\hat{\phi}(t)$ for $\phi(t)$ gives

$$\hat{\alpha} = \frac{\log \frac{\log|\hat{\phi}(t_1)|}{\log|\hat{\phi}(t_2)|}}{\log|\frac{t_1}{t_2}|}, \tag{4.16}$$

and

$$\log\hat{\sigma} = \frac{\log|t_1|\log(-\log|\hat{\phi}(t_2)|) - \log|t_2|\log(-\log|\hat{\phi}(t_1)|)}{\log|\frac{t_1}{t_2}|}. \tag{4.17}$$

In order to estimate β and μ define $u(t) \equiv Im(\log\phi(t))$. Then from (2.1)

$$u(t) = \mu t + \sigma^\alpha |t|^\alpha \beta sign(t) \tan\frac{\alpha\pi}{2}. \tag{4.18}$$

Choose two nonzero values of t, $t_3 \neq t_4$. Then

$$\frac{u(t_k)}{t_k} = \mu + \left[\sigma^\alpha |t|^{\alpha-1} \tan\frac{\alpha\pi}{2}\right]\beta, \tag{4.19}$$

for $k = 3, 4$. Since

$$\hat{\phi}(t) = \left(\frac{1}{n}\sum_{i=1}^n \cos(tx_i)\right) + i\left(\frac{1}{n}\sum_{i=1}^n \sin(tx_i)\right), \tag{4.20}$$

using elementary operations on complex numbers we get

$$\tan\hat{u}(t) = \frac{\sum\limits_{i=1}^n \cos(tx_i)}{\sum\limits_{i=1}^n \sin(tx_i)}. \tag{4.21}$$

Replacing $u(t)$ in (4.19) by the value $\hat{u}(t)$ from (4.21), and replacing the parameters by their estimators, and then solving the two linear equations simultaneously for β and μ gives the estimators

$$\hat{\beta} = \frac{\frac{\hat{u}(t_4)}{t_4} - \frac{\hat{u}(t_3)}{t_3}}{[|t_4|^{\hat{\alpha}-1} - |t_3|^{\hat{\alpha}-1}]\hat{\sigma}^{\alpha} \tan \frac{\hat{\alpha}\pi}{2}}, \tag{4.22}$$

and

$$\hat{\mu} = \frac{|t_4|^{\hat{\alpha}-1} \frac{\hat{u}(t_3)}{t_3} - |t_3|^{\hat{\alpha}-1} \frac{\hat{u}(t_4)}{t_4}}{|t_4|^{\hat{\alpha}-1} - |t_3|^{\hat{\alpha}-1}}. \tag{4.23}$$

Case of $\alpha = 1$. Under this condition from (4.15) we get a simpler estimate for σ

$$\hat{\sigma} = -\frac{\log |\hat{\phi}(t_1)|}{|t_1|}. \tag{4.24}$$

Using reasoning similar to the previous case, but for a different form of the cf we get

$$\hat{\beta} = \frac{\frac{\hat{u}(t_3)}{t_3} - \frac{\hat{u}(t_4)}{t_4}}{\frac{2}{\pi}\hat{\sigma} \log |\frac{t_4}{t_3}|}, \tag{4.25}$$

and

$$\hat{\mu} = \frac{\log |t_4| \frac{\hat{u}(t_3)}{t_3} - \log |t_3| \frac{\hat{u}(t_4)}{t_4}}{\log |t_4| - \log |t_3|}. \tag{4.26}$$

The estimators given above are consistent since they are based upon estimators of $\phi(t)$, $Im\phi(t)$ and $Re\phi(t)$, which are known to be consistent. However, convergence to the population values depends on the choice of $t_1, ..., t_4$. Optimal selection of these values is problematic. We discuss it in Section 4.4.

4.3.2 Minimum Distance Method

Two distributions are equal if and only if their respective cf agree over the real line. Basing on this fact, Press (1972a, 1972b) suggested two methods of estimation by means of the cf. In the *Minimum Distance Method* define

$$g(\alpha, \sigma, \beta, \mu) = \sup_t |\phi(t) - \hat{\phi}(t)|. \tag{4.27}$$

Then the minimum distance estimators of $(\alpha, \sigma, \beta, \mu)$ are the values that minimize (4.27).

A simple and computationally more accurate modification of the above is the *Minimum r–th Mean Distance Method*. In analogy to (4.27) define

$$h(\alpha, \sigma, \beta, \mu) = \int_{-\infty}^{\infty} |\phi(t) - \hat{\phi}(t)|^r W(t) dt, \tag{4.28}$$

where $W(t)$ denotes a suitable convergence factor (such that forces convergence of the integral). Then the minimum r–th mean distance estimators of $(\alpha, \sigma, \beta, \mu)$ are those values which minimize (4.28) for a fixed r. According to Press, both these methods yield consistent estimators, although he makes no claim of efficiency.

Leitch and Paulson (1975) and Paulson et al. (1975) were the first to construct estimators basing on the latter method. They have chosen $r = 2$ and $W(t) = \exp(-t^2)$, because of the computational advantage associated with Hermitian quadrature. Define $|\phi(t) - \hat{\phi}(t)|^2 = \lambda(t)$. The minimization of (4.28) may be accomplished numerically by a 20–point Hermitian quadrature, i.e.,

$$\int_{-\infty}^{\infty} \lambda(t) \exp(-t^2) dt = \sum_{k=1}^{20} w_k \lambda(u_k), \tag{4.29}$$

where u_k's are zeros of the Hermite polynomials and w_k's are weights associated with u_k's, respectively (see Abramowitz and Stegun (1964), page 924). α, σ, β and μ are assigned initial values and a gradient projection routine iterates to the optimal values of the estimates. For implementation details see Paulson et al. (1975).

We have not tested this method because of its computational complication. Moreover, a better, in terms of efficiency (see Koutrouvelis (1980)), and a simpler to implement method is available. We will discuss it in the next section.

4.3.3 Regression Method

Koutrouvelis (1980, 1981) presented a regression–type method of estimating the four parameters of a stable distribution. It is based on the following observations concerning the cf $\phi(t)$. First, from (2.1) we can easily derive

$$\log(-\log|\phi(t)|^2) = \log(2\sigma^\alpha) + \alpha \log|t|. \tag{4.30}$$

The real and imaginary parts of $\phi(t)$ are for $\alpha \neq 1$ given by

$$Re\phi(t) = \exp(-|\sigma t|^\alpha) \cos\left[\mu t + |\sigma t|^\alpha \beta sign(t) \tan\frac{\pi\alpha}{2}\right],$$

and

$$Im\phi(t) = \exp(-|\sigma t|^\alpha) \sin\left[\mu t + |\sigma t|^\alpha \beta sign(t) \tan\frac{\pi\alpha}{2}\right].$$

The last two equations lead, apart from considerations of principal values, to

$$\arctan\left(\frac{Im\phi(t)}{Re\phi(t)}\right) = \mu t + \beta\sigma^\alpha \tan\frac{\pi\alpha}{2} sign(t)|t|^\alpha. \tag{4.31}$$

Equation (4.30) depends only on α and σ and suggests that we estimate these parameters by regressing $y = \log(-\log|\phi_n(t)|^2)$ on $w = \log|t|$ in the model

$$y_k = m + \alpha w_k + \epsilon_k, \quad k = 1, 2, ..., K, \tag{4.32}$$

where (t_k) is an appropriate set of real numbers, $m = \log(2\sigma^\alpha)$, and ϵ_k denotes an error term. Koutrouvelis (1980) proposes to use $t_k = \frac{\pi k}{25}, k = 1, 2, ..., K$; K ranging between 9 and 134 for different estimates of the parameter α and sample sizes. Without significant loss of efficiency, we have simplified his method of choosing K. See Table 4.4.

Once $\hat{\alpha}$ and $\hat{\sigma}$ have been obtained and α and σ have been fixed at these values, estimates of β and μ can be obtained by using (4.31). Let $g_n(u) = Arctan(Im(\phi_n(u))/Re(\phi_n(u)))$, where $Arctan$ denotes the principal value of the arctan function. Then we can estimate β and μ by regressing $z = g_n(u) + \pi k_n(u)$ on u and $sign(u)|u|^\alpha$ in the model

$$z_l = \mu u_l + \beta\sigma^\alpha \tan\frac{\pi\alpha}{2} sign(u_l)|u_l|^\alpha + \eta_l, \quad l = 1, 2, ..., L, \tag{4.33}$$

index α	number K
1.5–2.0	10
0.6–1.5	20
0.4–0.6	60
less then 0.4	120

Table 4.4: *Number K of points ($t_k = \frac{\pi k}{25}; k = 1, ..., K$) for the model (4.32)*

where (u_l) is an appropriate set of real numbers and η_l denotes an error term. The integer $k_n(u)$ is introduced to account for possible nonprincipal branches of the arctan function. Koutrouvelis (1980) proposes to use $u_l = \frac{\pi l}{50}, l = 1, 2, ..., L$, L ranging between 9 and 70 for different estimates of the parameter α and sample sizes.

However, this regression model requires numerical inversion of matrices of size $n \times n$, where n is the size of the sample. On computers based on the 80x86 processor (all IBM PC compatibles), reasonably accurate algorithms for matrix inversion exist only for $n \leq 256$. Unfortunately, they are time consuming as well. These are the reasons for which we studied only the regression estimators of α and σ.

Now, we present a little modified regression method in easy to follow steps. In our simulations, the number of iterations was reduced to 1, but we give the algorithm in a recursive form. A limited simulation study indicated that the recursive scheme does not improve significantly the estimators after the first run. In addition, the simulations showed that it does not always converge.

Recursive algorithm for the regression–type estimation of a symmetric ($\beta = 0, \mu = 0$) stable distribution is as follows:

1. set the admissible error $0 < \epsilon < 1$ (accuracy of $\epsilon = 0.01$ will be usually reached in the first run) and the maximum number of iterations *max*;
2. find the initial estimates $\hat{\alpha}_0$ and $\hat{\sigma}_0$ using Fama–Roll's method ((4.5) and (4.4)), or using McCulloch's method (assuming $\beta = 0$);
3. set $\hat{\alpha} = \hat{\alpha}_0$ and $\hat{\sigma} = \hat{\sigma}_0$;
4. set new K depending on $\hat{\alpha}$, see Table 4.4;
5. scale the sample: $x_i = \frac{x_i}{\hat{\sigma}}$;
6. find the next estimate $\hat{a}$ of α

$$\hat{a} = \frac{\sum_{k=1}^{K} (w_k - \bar{w})(y_k - \bar{y})}{\sum_{k=1}^{K} (w_k - \bar{w})^2}; \tag{4.34}$$

7. find the next estimate $\hat{s}$ of σ from (4.30);
8. set $\hat{\alpha} = \hat{a}$ and $\hat{\sigma} = \hat{\sigma}\hat{s}$;

9. if $|\hat{s} - 1| \geq \epsilon$ and *the number of iterations* $<$ *max* then repeat steps 4 to 9 with new $\hat{\alpha}$ and $\hat{\sigma}$.

Finally, we obtain $\hat{\alpha}$ and $\hat{\sigma}$ which are regression–type estimates of α and σ, respectively.

4.4 Performance of the Estimators

The estimation procedures described in the previous sections: Fama–Roll's (FR), McCulloch's (CULL), moments (MOM) and regression (REG) were examined with the help of a simulation study. The random samples were generated using formula (3.3), combined with a linear transformation (3.5). For each desired set of parameter values, r samples of size n of stable random variables were generated. The uniform pseudo–random variables used in finding the stable ones were generated by Turbo Pascal's function *random*.

A series of simulation experiments were run first, with $\beta = 0$ and $\mu = 0$, and with a set of selected values for α and σ. The choice of these values was influenced by our desire to present as complete a picture of the estimation procedure as possible within our limited computer resources. Table 4.5 presents the results for four values of α and three values of σ, with $r = 25$ replications and sample size $n = 500$. The mean, minimum, maximum and Mean Squarred Error, i.e.

$$MSE_\theta = \frac{1}{n} \sum_{i=1}^{r} (\hat{\theta}_i - \theta)^2, \tag{4.35}$$

of the sampling distribution of α and σ are given.

The three estimation methods compared in Table 4.5 perform alike in the parameter space $\alpha \in [0.6, 2.0]$, $\sigma \in [0.1, 10]$:

- REG is a little better then both quantile methods when α is close to 2.0. This can be explained by the small size of the tails when the population distribution approaches the Gaussian;
- Quantile methods, especially CULL, are slightly better for $0.6 \leq \alpha \leq 1.0$. However, CULL cannot be used to estimate α below 0.6 and FR should not be used to estimate $\alpha \leq 1$. When α decreases below this level, FR becomes more and more inaccurate while REG retains its computational effectiveness;
- The CPU time needed to perform the REG estimation significantly increases as α decreases. This is caused by the larger value of K.

These simulations suggest to use REG as the most reliable estimator when the range of α and σ is unknown.

MOM is not included in Table 4.5, because of the enormous deviation from the true values when σ is not close to unity. The convergence to the population values depends on the choice of $t_1, ..., t_4$. Optimal selection of these values is problematic. Koutrouvelis (1980) uses $t_1 = .2, t_2 = .8, t_3 = .1, t_4 = .4$ in his comparison of the three sample cf methods. However, our studies proved only $t_1 = .2$ and $t_2 = .8$ to work satisfactorily and only in the symmetric case ($\beta = 0, \mu = 0$) with σ close to 1. We have tried using variable values depending on a prior estimate of σ for t_i's, but were not able to find satisfactory formulas. Moreover, this approach complicated the method and made it not so attractive from the computational point of view. Thus, in all simulations MOM uses $t_1 = .2$ and $t_2 = .8$.

Method	$\overline{\alpha}$	α_{min}	α_{max}	MSE_{α}	$\overline{\sigma}$	σ_{min}	σ_{max}	MSE_{σ}
$\alpha = 2.0, \sigma = 10$								
FR	1.926	1.717	2.000	0.01412	10.006	8.921	11.225	0.35238
CULL	1.937	1.689	2.000	0.01186	10.039	9.155	10.798	0.18676
REG	1.988	1.913	2.000	0.00053	10.077	9.240	10.772	0.11830
$\alpha = 1.5, \sigma = 10$								
FR	1.502	1.298	1.745	0.01128	10.230	8.989	11.291	0.35687
CULL	1.502	1.288	1.771	0.01491	10.100	9.073	11.095	0.33746
REG	1.486	1.245	1.628	0.00972	10.067	8.999	10.937	0.28761
$\alpha = 1.0, \sigma = 10$								
FR	1.010	0.902	1.150	0.00245	10.049	8.632	11.090	0.46313
CULL	1.013	0.872	1.219	0.00547	9.825	8.640	11.052	0.48814
REG	1.022	0.899	1.245	0.00551	10.069	8.729	10.946	0.46320
$\alpha = 0.6, \sigma = 10$								
FR	0.602	0.481	0.730	0.00213	10.603	8.213	12.766	2.15618
CULL	0.610	0.579	0.696	0.00127	9.371	7.968	12.425	1.86623
REG	0.621	0.507	0.745	0.00461	10.224	7.650	12.544	1.56676
$\alpha = 0.3, \sigma = 10$								
FR	0.330	0.299	0.381	0.00159	15.345	10.317	22.794	41.92433
REG	0.305	0.255	0.388	0.00091	10.835	7.591	15.358	6.34828
$\alpha = 2.0, \sigma = 1$								
FR	1.958	1.803	2.000	0.00493	0.988	0.913	1.094	0.00241
CULL	1.960	1.704	2.000	0.00607	0.990	0.895	1.095	0.00222
REG	1.993	1.947	2.000	0.00025	0.992	0.952	1.070	0.00108
$\alpha = 1.5, \sigma = 1$								
FR	1.518	1.351	1.677	0.00558	1.020	0.908	1.205	0.00447
CULL	1.518	1.415	1.692	0.00553	1.017	0.903	1.183	0.00394
REG	1.515	1.332	1.643	0.00729	1.011	0.914	1.148	0.00271
$\alpha = 1.0, \sigma = 1$								
FR	1.004	0.871	1.082	0.00254	1.015	0.864	1.190	0.00719
CULL	1.013	0.887	1.104	0.00259	1.007	0.840	1.156	0.00810
REG	0.999	0.854	1.079	0.00275	1.018	0.884	1.183	0.00661
$\alpha = 0.6, \sigma = 1$								
FR	0.606	0.552	0.760	0.00202	1.054	0.818	1.349	0.01839
CULL	0.603	0.566	0.744	0.00132	0.913	0.575	1.318	0.02782
REG	0.607	0.451	0.764	0.00430	0.989	0.767	1.202	0.00970
$\alpha = 0.3, \sigma = 1$								
FR	0.336	0.295	0.382	0.00215	1.457	0.866	2.478	0.32075
REG	0.305	0.262	0.350	0.00060	1.040	0.731	1.724	0.04911
$\alpha = 2.0, \sigma = 0.1$								
FR	1.950	1.754	2.000	0.00922	0.101	0.092	0.118	0.00004
CULL	1.955	1.722	2.000	0.00933	0.101	0.092	0.116	0.00003
REG	1.993	1.959	2.000	0.00020	0.100	0.092	0.107	0.00001
$\alpha = 1.5, \sigma = 0.1$								
FR	1.512	1.374	1.643	0.00595	0.099	0.088	0.110	0.00003
CULL	1.498	1.370	1.603	0.00437	0.098	0.087	0.108	0.00003
REG	1.495	1.361	1.620	0.00549	0.098	0.086	0.108	0.00004
$\alpha = 1.0, \sigma = 0.1$								
FR	1.003	0.857	1.114	0.00297	0.100	0.084	0.123	0.00008
CULL	0.993	0.793	1.091	0.00395	0.098	0.080	0.120	0.00009
REG	0.991	0.908	1.111	0.00309	0.100	0.084	0.116	0.00007
$\alpha = 0.6, \sigma = 0.1$								
FR	0.610	0.560	0.664	0.00095	0.105	0.080	0.124	0.00019
CULL	0.610	0.580	0.664	0.00071	0.094	0.074	0.119	0.00015
REG	0.609	0.525	0.712	0.00187	0.100	0.081	0.121	0.00011
$\alpha = 0.3, \sigma = 0.1$								
FR	0.327	0.298	0.387	0.00146	0.153	0.082	0.248	0.00416
REG	0.304	0.243	0.355	0.00064	0.106	0.068	0.142	0.00041

Table 4.5: *Performance table for Fama–Roll's (FR), McCulloch's (CULL) and regression (REG) estimators of* α *and* σ *for* $r = 25$ *and* $n = 500$.

Method	$\overline{\alpha}$	α_{min}	α_{max}	MSE_α	$\overline{\sigma}$	σ_{min}	σ_{max}	MSE_σ
				$r = 25, n = 2000, \alpha = 1.8$				
FR	1.812	1.726	1.950	0.00306	0.993	0.952	1.031	0.00055
CULL	1.817	1.714	2.000	0.00475	0.994	0.938	1.032	0.00054
MOM	1.800	1.745	1.856	0.00092	0.992	0.940	1.029	0.00038
REG	1.801	1.735	1.848	0.00101	0.992	0.948	1.023	0.00033
				$r = 25, n = 2000, \alpha = 1.5$				
FR	1.496	1.410	1.588	0.00260	1.003	0.946	1.047	0.00073
CULL	1.498	1.367	1.595	0.00288	1.001	0.931	1.051	0.00104
MOM	1.487	1.409	1.561	0.00208	0.999	0.953	1.048	0.00061
REG	1.492	1.371	1.630	0.00351	1.002	0.948	1.053	0.00082
				$r = 25, n = 2000, \alpha = 1.2$				
FR	1.207	1.159	1.264	0.00099	0.994	0.935	1.060	0.00082
CULL	1.211	1.135	1.275	0.00143	0.985	0.945	1.038	0.00096
MOM	1.204	1.111	1.276	0.00148	0.993	0.909	1.044	0.00095
REG	1.201	1.132	1.272	0.00122	0.992	0.927	1.071	0.00099
				$r = 25, n = 500, \alpha = 1.8$				
FR	1.794	1.597	1.989	0.00978	1.007	0.912	1.124	0.00316
CULL	1.806	1.597	2.000	0.01605	1.011	0.891	1.131	0.00378
MOM	1.776	1.560	1.909	0.00630	1.003	0.897	1.110	0.00243
REG	1.774	1.628	1.887	0.00528	1.003	0.895	1.108	0.00250
				$r = 25, n = 500, \alpha = 1.5$				
FR	1.497	1.329	1.733	0.00833	0.992	0.900	1.109	0.00285
CULL	1.487	1.274	1.690	0.01077	0.983	0.883	1.091	0.00320
MOM	1.501	1.277	1.698	0.00677	0.996	0.847	1.110	0.00389
REG	1.504	1.338	1.663	0.00664	0.991	0.894	1.147	0.00361
				$r = 25, n = 500, \alpha = 1.2$				
FR	1.174	1.063	1.353	0.00379	0.995	0.867	1.100	0.00442
CULL	1.169	1.041	1.329	0.00470	0.982	0.883	1.078	0.00420
MOM	1.155	1.017	1.299	0.00692	0.970	0.857	1.115	0.00459
REG	1.195	1.059	1.347	0.00614	1.000	0.870	1.143	0.00486
				$r = 50, n = 100, \alpha = 1.8$				
FR	1.779	1.217	2.000	0.03902	0.987	0.803	1.233	0.01037
CULL	1.788	1.284	2.000	0.04584	0.988	0.742	1.168	0.00957
MOM	1.828	1.427	2.000	0.02498	1.001	0.840	1.185	0.00756
REG	1.812	1.448	2.000	0.02528	0.995	0.824	1.182	0.00852
				$r = 50, n = 100, \alpha = 1.5$				
FR	1.497	1.140	2.000	0.04066	1.004	0.679	1.374	0.02051
CULL	1.504	1.113	2.000	0.04872	0.992	0.676	1.378	0.01980
MOM	1.527	1.116	1.958	0.04003	1.016	0.646	1.320	0.01970
REG	1.495	0.955	1.909	0.04424	0.998	0.654	1.313	0.01939
				$r = 50, n = 100, \alpha = 1.2$				
FR	1.165	0.887	1.453	0.02202	0.995	0.800	1.264	0.01196
CULL	1.150	0.870	1.460	0.02429	0.949	0.693	1.161	0.01371
MOM	1.198	0.848	1.564	0.02624	0.992	0.721	1.273	0.01654
REG	1.208	0.865	1.543	0.01851	1.004	0.754	1.289	0.01255

Table 4.6: *Performance table for Fama–Roll's (FR), McCulloch's (CULL), moments (MOM) and regression (REG) estimators of* $\alpha = 1.8, 1.5, 1.2$ *and* $\sigma = 1$ *for different sample sizes* n *and a different number of replications* r.

Table 4.6 presents the performance of all four estimation methods for different sample sizes. When $\alpha \in [1.2, 1.8]$ and $\sigma = 1$ there are no significant differences between these methods:

- REG performs slightly better for all sample sizes - its Mean Squarred Errors of α and σ are in general the lowest. For $n = 2000$ and $r = 25$, the error between the population values and REG estimates is less then 0.01;

- Estimation of parameters in samples of small size, e.g. $n = 100$, is possible only for a moderate number of replications. $r = 50$ is enough, at least for REG. Single estimators of α may yield errors as large as 0.4.

These simulations suggest to use REG as the most reliable estimator for different sample sizes. However, when we believe that σ is close to unity and $\alpha > 1$, so that we can use quantile methods, all four algorithms give relatively good estimators and we can base our choice on the CPU time needed. MOM is the fastest, but we recommend to use CULL as it is fast and performs well for $\alpha \geq 0.6$ and all admissible values of σ. Moreover, as the only method compared in this paper, it gives estimators of β. In Table 4.7 we present the behavior of $\hat{\beta}$. It performs best in the middle of the range of α. When α is close to 2.0 it underestimates large values of β ($|\beta| > 0.7$). On the other hand, when α is close to 0.6 it overestimates large values of β.

β	$\overline{\alpha}$	α_{min}	α_{max}	MSE_α	$\overline{\beta}$	β_{min}	β_{max}	MSE_β
				$\alpha = 1.8$				
0.00	1.771	1.639	2.000	0.01036	0.074	-0.987	1.000	0.17234
0.25	1.858	1.629	2.000	0.01709	0.253	-0.442	1.000	0.14051
0.50	1.825	1.564	2.000	0.01622	0.457	-0.174	1.000	0.15024
0.75	1.822	1.618	2.000	0.01324	0.563	-0.276	1.000	0.19610
1.00	1.810	1.524	2.000	0.02024	0.817	0.000	1.000	0.09915
				$\alpha = 1.3$				
0.00	1.324	1.202	1.535	0.00640	-0.016	-0.316	0.391	0.02706
0.25	1.295	1.104	1.412	0.00708	0.280	0.019	0.508	0.01409
0.50	1.338	1.136	1.692	0.01506	0.582	0.116	1.000	0.03247
0.75	1.297	1.134	1.529	0.01285	0.838	0.547	1.000	0.02500
1.00	1.302	1.071	1.557	0.01292	0.976	0.864	1.000	0.00214
				$\alpha = 0.8$				
0.00	0.793	0.718	0.868	0.00162	-0.020	-0.266	0.245	0.01449
0.25	0.795	0.722	0.937	0.00264	0.241	0.052	0.507	0.01369
0.50	0.814	0.654	0.955	0.00676	0.591	0.364	0.982	0.03989
0.75	0.821	0.674	1.055	0.00567	0.985	0.796	1.000	0.05734
1.00	0.811	0.685	1.036	0.00745	1.000	1.000	1.000	0.00000

Table 4.7: *Performance table for McCulloch's estimator of* $\alpha = 1.8, 1.3, 0.8$ *and* $\beta = 0.0, 0.25, 0.5, 0.75, 1.0$ *for sample size* $n = 500$ *and* $r = 25$ *replications.*

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